8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that it must hold when $n = m + 1$.

Suggestion: When $n = m + 1$, write

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)(z_1 + z_2)^m = (z_2 + z_1)\sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m-k}$$

$$= \sum_{k=0}^{m}\binom{m}{k} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m}\binom{m}{k} z_1^{k+1} z_2^{m-k}$$

and replace k by $k-1$ in the last sum here to obtain

$$(z_1 + z_2)^{m+1} = z_2^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] z_1^k z_2^{m+1-k} + z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

Sol : • When $n = 1$,

$$(z_1 + z_2)^n = z_1 + z_2 = \sum_{k=0}^{1} C_k^1 z_1^k z_2^{m-k}$$

• Assume it is true for $n = m$.

• When $n = m+1$,

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)^m (z_1 + z_2)$$

$$= \left(\sum_{k=0}^{m} C_k^m z_1^k z_2^{m-k}\right)(z_1 + z_2)$$

$$= \sum_{k=0}^{m} C_k^m z_1^{k+1} z_2^{m-k} + \sum_{k=0}^{m} C_k^m z_1^k z_2^{m+1-k}$$

$$= z_1^{m+1} + \sum_{k=1}^{m} \left(C_k^m + C_{k-1}^m \right) z_1^k z_2^{m+1-k} + z_2^{m+1}$$

$$= z_1^{m+1} + \sum_{k=1}^{m} C_k^{m+1} z_1^k z_2^{m+1-k} + z_2^{m+1}$$

$$= \sum_{k=0}^{m+1} C_k^{m+1} z_1^k z_2^{m+1-k}$$

(Note that $C_k^m + C_{k-1}^m = \dfrac{m!}{k!(m-k)!} + \dfrac{m!}{(k-1)!(m-k+1)!}$

$$= \frac{m!(k + m+1-k)}{k!(m+1-k)!}$$

$$= \frac{(m+1)!}{k!(m+1-k)!}$$

$$= C_k^{m+1} \quad)$$

By M.I., the formula is true for any $n \in \mathbb{N}$.

$\square$

4. Verify that $\sqrt{2}\,|z| \geq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$.

Suggestion: Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

Sol : $(|x| - |y|)^2 \geq 0$

$\Leftrightarrow |x|^2 - 2|x||y| + |y|^2 \geq 0$

$\Leftrightarrow 2(x^2+y^2) \geq (|x|+|y|)^2$ (Rmk: This is Cauchy's Inequality)

$\Leftrightarrow \sqrt{2}\sqrt{x^2+y^2} \geq |x| + |y|$

Write $z = x+iy$, we are done.

$\square$

6. Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that $|z-1| = |z+i|$ represents the line through the origin whose slope is -1.

Sol: 'A point z satisfies $|z-1| = |z+i|$'

is equivalent to

'z is equidistant from 1 and $-i$'.

Then z is on the perpendicular bisector of the segment joining 1 and $-i$, which is the line through the origin whose slope is -1.

$\square$

8. Let z_1 and z_2 denote any complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2.$$

Use simple algebra to show that

$$|(x_1 + iy_1)(x_2 + iy_2)| \quad \text{and} \quad \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

are the same and then point out how the identity

$$|z_1 z_2| = |z_1||z_2|$$

follows.

Sol:

$$
\begin{aligned}
& |(x_1 + iy_1)(x_2 + iy_2)|^2 \\
&= |x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)|^2 \\
&= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2 \\
&= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 \\
&= x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2) \\
&= (x_1^2 + y_1^2)(x_2^2 + y_2^2).
\end{aligned}
$$

Hence, $|(x_1 + iy_1)(x_2 + iy_2)| = \sqrt{(x_1^2 + y_2^2)(x_2^2 + y_2)^2}$.

Take $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we have

$$|z_1 z_2| = |z_1||z_2|.$$

$\square$

1. Use properties of conjugates and moduli established in Sec. 6 to show that

(*a*) $\overline{\bar{z} + 3i} = z - 3i$; (*b*) $\overline{iz} = -i\bar{z}$;

(*c*) $\overline{(2+i)^2} = 3 - 4i$; (*d*) $|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$.

Sol: (a) $\overline{\bar{z}+3i} = \bar{\bar{z}} + \overline{3i} = z - 3i$

(b) $\overline{iz} = \bar{i}\,\bar{z} = -i\bar{z}$

(c) $\overline{(2+i)^2} = (\overline{2+i})^2 = (2-i)^2 = 3-4i$

(d) $|(2\bar{z}+5)(\sqrt{2}-i)| = |2\bar{z}+5|\,|\sqrt{2}-i|$

$= \sqrt{3}\,|\overline{2\bar{z}+5}|$

$= \sqrt{3}\,|2z+5|$

$\square$

9. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (2), Sec. 5, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

Sol: Write $z^4 - 4z^2 + 3 = (z^2-1)(z^2-3)$.

When $|z| = 2$,

$$|z^2 - 1| \geq |z|^2 - 1 = 3.$$

$$|z^2 - 3| \geq |z|^2 - 3 = 1.$$

Then $|z^4 - 4z^2 + 3| = |z^2-1|\,|z^2-3|$

$$\geq 3 \times 1 = 3.$$

Hence, $\left|\dfrac{1}{z^4-4z^2+3}\right| = \dfrac{1}{|z^4-4z^2+3|} \leq \dfrac{1}{3}$.

$\square$

5. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(*a*) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; (*b*) $5i/(2+i) = 1+2i$;

(*c*) $(\sqrt{3}+i)^6 = -64$; (*d*) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

Sol: (a) $i(1-\sqrt{3}i)(\sqrt{3}+i)$

$$= e^{\frac{\pi}{2}i}\left(2e^{-\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{6}i}\right)$$

$$= 4e^{\frac{\pi}{3}i}$$

$$= 4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$= 2(1+\sqrt{3}i)$$

(b) $\dfrac{5i}{2+i} = \dfrac{5e^{\frac{\pi}{2}i}}{\sqrt{5}e^{i\theta}}$ where $\begin{cases} \cos\theta = \frac{2}{\sqrt{5}} \\ \sin\theta = \frac{1}{\sqrt{5}} \end{cases}$

$$= \sqrt{5}\, e^{\left(\frac{\pi}{2}-\theta\right)i}$$

$$= \sqrt{5}\left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)$$

$$= \sqrt{5}(\sin\theta + i\cos\theta)$$

$$= 1+2i$$

(c) $(\sqrt{3}+i)^6 = \left(2e^{\frac{\pi}{6}i}\right)^6$

$$= 2^6 \cdot e^{\pi i}$$

$$= -64$$

(d) $(1+\sqrt{3}i)^{-10} = \left(2e^{\frac{\pi}{3}i}\right)^{-10}$

$$= 2^{-10} e^{-\frac{10}{3}\pi i}$$

$$= 2^{-10} e^{\frac{2}{3}\pi i}$$

$$= 2^{-10}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$= -2^{11}(-1+\sqrt{3}i)$$

$\square$

9. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive ***Lagrange's trigonometric identity:***

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

Suggestion: As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

Sol: Let $S = 1 + z + \cdots + z^n$.

Then $zS = z + z^2 + \cdots + z^{n+1}$.

Then $(1-z)S = 1 - z^{n+1}$.

When $z \neq 1$, $S = \dfrac{1 - z^{n+1}}{1 - z}$

Let $z = e^{i\theta}$.

On the one hand,

$1 + z + \cdots + z^n = 1 + e^{i\theta} + \cdots + e^{in\theta}$

$= (1 + \cos\theta + \cdots + \cos n\theta) + i(\sin\theta + \cdots + \sin n\theta)$

On the other side, $\dfrac{1 - z^{n+1}}{1 - z} = \dfrac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$

$$= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

$$= \frac{e^{i\frac{2n+1}{2}\theta} - e^{-\frac{i\theta}{2}}}{e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}}$$

$$= \frac{-\frac{i}{2}\left(e^{i\frac{2n+1}{2}\theta} - e^{-\frac{i}{2}\theta}\right)}{\frac{e^{\frac{i}{2}\theta} - e^{-\frac{i}{2}\theta}}{2i}}$$

$$= \frac{-i}{2\sin\frac{\theta}{2}}\left(e^{i\frac{2n+1}{2}\theta} - e^{-\frac{i}{2}\theta}\right)$$

(Rmk: Since $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$,

then $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$)

Hence, $1 + \cos\theta + \cdots + \cos(n\theta) = \mathrm{Re}\left(\frac{1 - z^{n+1}}{1 - z}\right)$

$$= \frac{1}{2\sin\frac{\theta}{2}}\left(\sin\frac{2n+1}{2}\theta + \sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}. \qquad \square$$

10. Use de Moivre's formula (Sec. 8) to derive the following trigonometric identities:

(*a*) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$;

(*b*) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

Sol: Note that $e^{3i\theta} = (e^{i\theta})^3$

On the one hand,

$$e^{3i\theta} = \cos 3\theta + i\sin 3\theta.$$

On the other hand,

$$(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$$

$$= [(\cos^2\theta - \sin^2\theta) + 2i\sin\theta](\cos\theta + i\sin\theta)$$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

(a). Take real part, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

(b). Take imaginary part, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

1. Find the square roots of (a) $2i$; (b) $1-\sqrt{3}i$ and express them in rectangular coordinates.

Ans. (a) $\pm(1+i)$; (b) $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

Sol: (a) $2i = 2e^{\frac{\pi}{2}i}$

$$= \left(\pm\sqrt{2}e^{\frac{\pi}{4}i}\right)^2$$

$$= \left(\pm(1+i)\right)^2$$

The square roots of $2i$ are $\pm(1+i)$

(b) $1-\sqrt{3}i = 2e^{\frac{2}{3}i}$

$$= \left(\pm\sqrt{2}e^{\frac{1}{3}i}\right)^2$$

$$= \left(\pm\sqrt{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)^2$$

$$= \left(\pm\frac{1+\sqrt{3}i}{\sqrt{2}}\right)^2$$

The square roots of $1-\sqrt{3}i$ are $\pm\dfrac{1+\sqrt{3}i}{\sqrt{2}}$.

$\square$

8. (*a*) Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \qquad (a \neq 0)$$

when the coefficients a, b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the ***quadratic formula***

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$,

(*b*) Use the result in part (*a*) to find the roots of the equation $z^2 + 2z + (1 - i) = 0$.

Ans. (*b*) $\left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}, \quad \left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}.$

Sol: (a) Note that when $a \neq 0$,

$$az^2 + bz + c = a\left(z^2 + \frac{b}{a}z\right) + c$$

$$= a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Then $az^2 + bz + c = 0$

$$\Longleftrightarrow \quad \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Longleftrightarrow \quad z = \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

(Rmk: Here, $(b^2 - 4ac)^{\frac{1}{2}}$ is not unique in general.)

(b) The roots of the equation $z^2+2z+(1-i)=0$

is $$z = \frac{-2+(4-4(1-i))^{\frac{1}{2}}}{2}$$

$$= \frac{-2+2\,i^{\frac{1}{2}}}{2}$$

$$= -1+(e^{\frac{\pi}{2}i})^{\frac{1}{2}}$$

$$= -1 \pm e^{\frac{\pi}{4}i}$$

$$= -1 \pm \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$

$$= \left(-1+\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}i \quad \text{or} \quad \left(-1-\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{2}}{2}i$$

$\square$

5. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z-2| < 1$. State why S is not connected.

Sol: Let S_1 be the set consisting of all points z such that $|z|<1$.

Let S_2 be the set consisting of all points z such that $|z-2|<1$.

Clearly, S_1, S_2 are open and $S_1 \cup S_2 = S$.

To show S is not connected, it suffices to show $S_1 \cap S_2 = \phi$.

· If $z \in S_2$, then $|z-2|<1$. Then

$$|z| = |z-2+2| \geq 2-|z-2| > 2-1 = 1$$

Therefore, $z \notin S_1$. Hence, $S_1 \cap S_2 = \phi$.

$\square$

8. Prove that if a set contains each of its accumulation points, then it must be a closed set.

Sol: Let S be a set in $\mathbb{C}$ which contains all its accumulation points.

Let ∂S be the boundary of S.

Suppose $a \in \partial S$. Then $\forall \varepsilon > 0$, $\exists z \in S$ s.t. $|z-a| < \varepsilon$.

Then a is an accumulation point of S.

Therefore, $\partial S \subset S$.

Hence, S is closed.

$\square$

4. Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r,\theta) + iv(r,\theta)$.

Ans. $f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$.

Sol: Let $z = re^{i\theta}$.

Then
$$\begin{aligned} f(z) &= z + \frac{1}{z} \\ &= re^{i\theta} + \frac{1}{re^{i\theta}} \\ &= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

$\square$